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Parameterizations for 3PL and GPC Models: A Common Alternative and Method in flexMIRT®/ IRTPRO™

Introduction

flexMIRT® and IRTPRO™ use different parameterizations for common IRT models in order to implement the general multilevel IRT (flexMIRT®) and multidimensional IRT (flexMIRT® and IRTPRO™) modeling frameworks. This note presents the different parameterizations for the three-parameter logistic (3PL) and generalized partial credit (GPC) model, and is based almost entirely on Thissen, Cai, and Bock (2010).

Notation

For an item, let there be m categories, with category response $k \in \{0, \dots, m-1\}$. To avoid confusion, parameters in the alternative parameterization will be denoted with an asterisk (*). The scaling constant $D = 1.702$.

Models:

1. 3PL

Alternative

$$P(y=1|\theta) = g + \frac{1-g}{1+\exp\left(-Da^*(\theta - b^*)\right)} \tag{1}$$

flexMIRT or IRTPRO

$$P(y=1|\theta) = g(z) + \frac{1-g(z)}{1+\exp\left(-(a\theta + c)\right)} \tag{2}$$

The following conversions can be used to convert from (1) to (2):

$$a = Da^*, \qquad c = -Da^*b^* \tag{3}$$

To convert from (2) to (1), solve as needed. In flexMIRT® or IRTPRO™, logit-guessing is the estimated parameter:

$$g(z) = \frac{1}{1+\exp(-z)}. \tag{4}$$

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2. GPC

Alternative

$$P(y=k|\theta)=\frac{\exp\left(\sum_{j=0}^{k}Da^*(\theta-b^*+d_j^*)\right)}{\sum_{i=0}^{m-1}\exp\left(\sum_{j=0}^{i}Da^*(\theta-b^*+d_j^*)\right)}, \quad (5)$$

where $\boldsymbol{d}^*$ is an $m \times 1$ vector of step parameters, $d_1^* = 0$ and $\sum_{j=2}^{m} d_j^* = 0$. Thus, there are m free parameters (a^*, b^*, and $m-2$ step parameters).

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$$P(y=k|\theta)=\frac{\exp(z_k)}{\sum_{i=0}^{m-1}\exp(z_i)}, \quad (6)$$

where

$$z_k = \breve{a}a_{k+1}^{s}\theta + c_{k+1}, \quad (7)$$

and $\breve{a}$ is the overall slope parameter, a_{k+1}^{s} is the scoring function for response k, and c_{k+1} is the intercept parameter. Equation (6) is the general nominal model. Instead of estimating the under-identified elements of $\boldsymbol{a^s}$ and $\boldsymbol{c}$ (which are $m \times 1$ vectors), flexMIRT estimates elements in $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ (which are $(m-1) \times 1$ vectors) by reparameterization, as well as $\breve{a}$. The two sets of vectors are related by:

$$\boldsymbol{a^s} = \mathbf{T}\boldsymbol{\alpha}, \text{ and } \boldsymbol{c} = \mathbf{T}\boldsymbol{\gamma} \quad (8)$$

and $\mathbf{T}$ is an $m \times (m-1)$ contrast coefficient matrix. Both flexMIRT® and IRTPRO™ implement the Fourier contrasts and the Identity contrasts described by Thissen, Cai, and Bock (2010). In addition, flexMIRT® offers direct user-defined control over the contrast coefficient matrix.

For the GCP model, $\alpha_1 = 1$, and $\alpha_2 \ldots \alpha_{m-1} = 0$, and these are not estimated. Thus, there are m free parameters ($\breve{a}$, and $m-1$ parameters in $\boldsymbol{\gamma}$).

To convert from (6) to flexMIRT® or IRTPRO™ parameters, use:

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$$\breve{a} = Da^*, \tag{9}$$

$\boldsymbol{\alpha}$ is as described above, and $\boldsymbol{\gamma}$ depends on $\boldsymbol{c}$. The first element, $c_1 = 0$, and for $j = 2, \dots, m$,

$$c_j = (d^*_{j-1} - b^*)Da^* + c_{j-1}. \tag{10}$$

Then for any contrast coefficient matrix $\mathbf{T}$, $\boldsymbol{\gamma}$ is obtained as

$$\boldsymbol{\gamma} = (\mathbf{T}'\mathbf{T})^{-\mathbf{1}}\mathbf{T}'\boldsymbol{c}. \tag{11}$$

To convert from flexMIRT® or IRTPRO™ parameters to parameters in (5), use:

$$a^* = \frac{\breve{a}}{D}, \qquad b^* = -\frac{\gamma_1}{\breve{a}}, \tag{12}$$

and $\boldsymbol{d}^*$ depends on $\boldsymbol{c} = \mathbf{T}\boldsymbol{\gamma}$. As mentioned above, $d^*_1 = 0$, and for $j = 2, \dots, m$,

$$d^*_j = \frac{c_j - c_{j-1}}{\breve{a}} + b^*. \tag{13}$$